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BOMB TRAJECTORIES

NATIONAL ADVISORY COMMITTEE
FOR AERONAUTICS

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I. INTRODUCTORY.

The trajectory of a bomb of high terminal velocity dropped from a great altitude such as 30,000 feet requires a complicated analysis much like that for the trajectory of a shell fired at a high angle. For, in the first place, the changing density of the air can not be wholly ignored, and in the second place, the air resistance as a function of the velocity is exceedingly complicated, and, in particular, as the bomb may pass through the velocity of sound in air, all sorts of complications leading even to instability and tumbling, which render any calculation of the trajectory illusory, may be present.

I have pointed out in a previous note on the Limiting Velocity of Free Fall that bombs of high terminal velocity must fall over 10,000 feet and acquire a velocity of 800 ft./sec., that bombs of lower terminal velocity must fall through even greater distances, and that there is tolerable agreement among ballisticians that up to velocities of about 800 ft./sec., the simple square law of resistance ($R \propto v^2$) holds for solid shell-like bodies. The problem which I wish to treat here is that of the trajectory of a bomb launched horizontally when the fall is not so great that the velocity exceeds 800 ft./sec., at any point of the path. Only the first approximation founded on the assumption of constant air density will be developed.

II. GENERAL CONSIDERATIONS.

This problem has been treated in several ways in an elementary manner, e. g., by E. L. Gayhart in "Aviation," Volume III, No. 12, January 15, 1918, pages 819–822, after the German, by Ernest Hemkel. My discussion, likewise elementary, is that which I have given students of aeronautical engineering at the Massachusetts Institute of Technology, and which may be of interest to others.

If V be the velocity in path, u, v the component velocities horizontal and vertical downward, respectively, U the terminal velocity, the resistance is $R=kV^2=WV^2/U^2$ pounds, where W is the mass. The equations of motion are

$$\frac{du}{dt}=-g\frac{uV}{U^2}, \quad \frac{dv}{dt}=g\left(1-\frac{vV}{U^2}\right) \tag{1}$$

or

$$\frac{V^2}{r}=g\frac{u}{V}, \quad \frac{dV}{dt}=g\left(\frac{v}{V}-\frac{V^2}{U^2}\right), \tag{2}$$

according as the motion is referred to rectangular axes or to the tangent and normal to the path (radius of curvature$=r$).

The second of (2) may be written

$$V\frac{dV}{dy}=g\left(1-\frac{V^2}{U^2}\;\frac{V}{v}\right),$$

and compared with the equation for vertical fall in which $V=v$. For the trajectory V/v is always greater than unity and hence the increase of V with the vertical drop y is always less than the increase of v with vertical drop y when the fall is itself vertical. In other words, in the trajectory the bomb gains tangential velocity with vertical drop more slowly than vertical velocity is gained in free fall.

Now bombs are released with a horizontal velocity $V_0 = u_0$ which is small relative to their terminal velocity U. Even a fast-moving airplane will hardly exceed 200 ft./sec., and a bomb with low terminal velocity will have U probably not less than 800 ft./sec., so that u_0/U is almost sure not to exceed 1/4. In the case of free fall

$$\frac{2gh}{U^2} = \log_e\left(1 - \frac{v_0^2}{U^2}\right) - \log_e\left(1 - \frac{v^2}{U^2}\right)$$

and the distance h of fall required to reach a velocity of v from rest exceeds that required to reach the same velocity from an initial vertical velocity of $v_0 = U/4$ by only the amount

$$\Delta h = \frac{-U^2}{2g}\log_e\left(1 - \frac{1}{16}\right) = \frac{U^2}{32g} = \left(\frac{U}{g}\right)^2, \text{ approx.}$$

If U be 800 ft./sec., the additional distance is only 625 feet.

It may therefore be stated that a bomb launched horizontally with a velocity as high as 200 ft./sec. and with a terminal velocity as low as 800 ft./sec. will require, to attain a linear (tangential) velocity of V, a vertical drop of less than 625 feet less than that required by a body falling from rest. (If v_0/U is, as it generally is, decidedly smaller than 1/4, the distance 625 feet is very much reduced.)

The reasoning shows that for all practical purposes one may consider that a velocity on a trajectory will remain under 800 ft./sec. even though the initial velocity be high, provided the vertical drop is not so large as to generate a vertical velocity in excess of 800 ft./sec in the case of free fall. This brings the safety limit for the application of the square law for resistance back practically to the case of the previous note.

The relation between the arc s described on the trajectory and the velocity acquired may be discussed from the second of (2).

For

$$V\frac{dV}{ds} = g\left(\frac{v}{V} - \frac{V^2}{U^2}\right)$$

may be compared with

$$v\frac{dv}{dy} = g\left(1 - \frac{v^2}{U^2}\right)$$

to see that tangential velocity V is gained along the trajectory relative to the distance traveled, s, much more slowly than vertical velocity is gained relative to y in the case of straight fall, the term v/V being always less than 1. In fact, at the start, tangential velocity is lost, since $v/V = 0$.

III. PRELIMINARY INTEGRATION.

With reference to the arc s of the trajectory the first equation of (1) may be integrated. Thus—

$$\frac{du}{dt} = \frac{du}{ds}\,\frac{ds}{dt} = V\frac{du}{ds}, \quad \frac{du}{ds} = -\frac{gu}{U^2}, \quad u = u_0 e^{-gs/U^2}. \tag{3}$$

The horizontal velocity falls off exponentially with the arc traveled. For example, if $U = 800$, $g/U^2 = 1/20{,}000$; the horizontal velocity will be reduced 1/2 only after a travel of 14,000 feet. Now if the time to be eliminated between the two equations (1) to obtain the differential equation of the trajectory, or, better, if the first equation of (2) be used for this purpose, it is seen that

$$\frac{d^2y}{dx^2} = \frac{g}{u^2} = \frac{g}{u_0^2}e^{2gs/U^2}. \tag{4}$$

For

$$\frac{1}{r} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} = \frac{\dfrac{d^2y}{dx^2}}{\left(1 + \dfrac{v^2}{u^2}\right)^{3/2}} = \frac{\dfrac{d^2y}{dx^2}}{\dfrac{V^3}{u^3}} = g\frac{u}{V^3}$$

(This shows that the equation $\frac{d^2y}{dx^2}=\frac{g}{u^2}$ holds for *any* law of resistance.) Hence the second derivitive of y by x increases inversely as the square of the horizontal velocity or increases exponentially with the arc of the trajectory. For the parabola followed by the bomb in vacuo $U=\infty$ and

$$\frac{d^2y}{dx^2}=\frac{g}{u_0^2} \tag{4'}$$

Inasmuch as

$$\frac{dy}{dx}=\int^x \frac{d^2y}{dx^2}\,dx, \quad y=\int^x\left[\int^x \frac{d^2y}{dx^2}dx\right]dx$$

and since $e^{2gs/U^2}>1$, it follows at once from (4) and (4′) that, for a given horizontal travel x, a bomb will fall farther in the air than in vacuo—or, for a given drop h, a bomb will fall short in the air as compared with its position in vacuo. (This would seem to be self-evident, but Hemkel, as translated by Gayhart in "Aviation," cit. sup., states that "the paradox may arise that because of resistance the bomb will travel farther in the air than if in a vacuum"—a remark that I do not understand.)

IV. THE PARABOLA.

If axes be taken at the starting point, the trajectory in vacuo is the parabola

$$y=gx^2/2u_0^2=1/2\ mx^2 \text{ if } m=g/u_0^2.$$

This is the first approximation for the trajectory.

The arc of the parabola $y=1/2\ mx^2$ may be obtained as a series, namely,

$$s=\int^x \sqrt{1+m^2x^2}dx=\int (1+1/2\ m^2x^2+\ldots.)\ dx=x+1/6\ m^2x^3+\ldots..$$

The approximation is good as long as mx is small relative to 1; it becomes bad as mx nears 1, and for $mx>1$ the series diverges. Now $m=g/u_0^2$. If u_0 is as high as 180 ft./sec., m is as small as 1/1000, and the horizontal travel may be several hundred feet before s differs much from x. If, however, the airplane be moving at only 115 ft./sec., $u_0^2=13{,}200$, $m=g/u_0^2=1/400$, and the horizontal travel can hardly exceed 300 feet before s becomes considerably different from x. As mx is the quantity determining the degree of approximation, the equation $y=1/2\ mx^2$ may best be written as $y=(mx)x/2$, from which it is seen that the drop in vacuo can only be a few hundred feet at best if mx is to remain small. For these small drops the trajectory in air does not depart appreciably from that in vacuo.

V. THE SECOND APPROXIMATION.

The departure from the parabola for moderate drops may be calculated by integrating (4) with $s=x$.

$$\frac{d^2y}{dx^2}=\frac{g}{u_0^2}e^{2gx/U^2}=\frac{g}{u_0^2}\left(1+\frac{2gx}{U^2}\right),\ y=\frac{gx^2}{2u_0^2}\left(1+\frac{2gx}{3\,U^2}\right), \text{ approx.} \tag{5}$$

The expansion may be reverted so that x appears as a series in y 1/2.

$$x=u_0\sqrt{\frac{2y}{g}}\left(1+\frac{2gx}{3\,U^2}\right)^{-1/2}=u_0\sqrt{\frac{2y}{g}}\left(1-\frac{gx}{3\,U^2}\right),\ x=u_0\sqrt{\frac{2y}{g}}\left(1-\frac{gu_0}{3\,U^2}\sqrt{\frac{2y}{g}}\right)=u_0\sqrt{\frac{2y}{g}}-\frac{2u_0^2y}{3\,U^2}. \tag{6}$$

If $u_0/U=1/4$, as might be the case for a high-speed machine and a bomb of low terminal velocity, the correction shows that the bomb in air will fall short of the parabolic position by $-\Delta x=y/24$. As the approximation is only good for values of y running to a few hundred feet, the error in using the parabola is only some 20 feet at most. In case of a slow machine $(u_0=100)$ and a high terminal velocity $(U=1{,}500)$ the correction would be under 2 feet.

VI. A THIRD APPROXIMATION.

The formula (6) for the horizontal carry is not surely valid for values of y as large as 1,000 feet. It becomes necessary to seek a better integral of (4). The most direct method would be to expand y into a Maclaurin's series by repeated differentiation of (4). The initial conditions are

$$x=0,\ y=0,\ dy/dx=0,\ s=0 \text{ and } d^2y/dx^2=g/u_0^2,$$

$$\frac{d^3y}{dx^3}=\frac{g}{u_0^2}\cdot\frac{2g}{U^2}e^{2gs/U^2}\sqrt{1+\left(\frac{dy}{dx}\right)^2},\quad \left(\frac{d^3y}{dx^3}\right)_0=\frac{g}{u_0^2}\cdot\frac{2g}{U^2}.$$

From this point on the differentiations become involved. Every time the exponential is differentiated the very small factor $2g/U^2$ is introduced, but with the differentiation of the radical the factor g/u_0^2, which is not so small, is had. The results are

$$\left(\frac{d^4y}{dx^4}\right)_0=\frac{g}{u_0^2}\left(\frac{2g}{U^2}\right)^2,\quad \left(\frac{d^5y}{dx^5}\right)_0=\frac{g}{u_0^2}\cdot\frac{2g}{U^2}\left[\left(\frac{2g}{U^2}\right)^2+\left(\frac{g}{u_0^2}\right)^2\right]$$

The second term in the fifth derivative is large compared with the first if u_0^4/U^4 is a small number—and it is almost always very small. The series for y then becomes

$$y=\frac{gx^2}{2u_0^2}\left(1+\frac{2gx}{3U^2}+\frac{g^2x^2}{3U^4}+\frac{gx}{30U^2}\frac{g^2x^2}{u_0^4}+\ ---\right). \tag{7}$$

The occurrence of the term $(gx/u_0^2)^2$, with the repetition of similar terms in higher powers, makes it clear that not gx/U^2 but gx/u_0^2 is the number which must be kept small if the series is to converge rapidly and be valuable. Now, with reference to the parabolic (first) approximation, $gx/u_0^2=2y/x$; and hence it is inferred that the series expansion (7) is not valid except when $2y/x$ is not large, i. e., when y is not more than a few hundred feet. In fact, a comparison shows that the fourth term in (7) is equal to the second term, which is identical with the correction in (5), when $gx/u_0^2=4.5$. In the case of a machine for which $u_0=100$, $x=1{,}400$. If (7) held good for such large values of x, it would be valid up to drops of some 3,000 feet. This would be very satisfactory, but there is no assurance that the subsequent terms in (7) will be small relative to those already obtained when gx/u_0^2 is as great as 4.5.

However, if the third term of (7) be discarded as small relative to the second, the approximation is

$$\begin{aligned} y&=\frac{gx^2}{2u_0^2}\left(1+\frac{2gx}{3U^2}+\frac{gx}{30U^2}\frac{g^2x^2}{u_0^4}\right)\\ x&=u_0\sqrt{\frac{2y}{g}}\left(1-\frac{gx}{3U^2}-\frac{gx}{60U^2}\frac{g^2x^2}{u_0^4}\right)\\ x&=u_0\sqrt{\frac{2y}{g}}-\frac{2u_0^2y}{3U^2}-\frac{gy^2}{15U^2}\end{aligned} \tag{8}$$

The correction for the carry x relative to the parabolic path is

$$-\Delta x=\frac{u_0^2y}{3U^2}+\frac{gy^2}{30U^2}=\frac{2u_0^2y}{3U^2}\left(1+\frac{gy}{10u_0^2}\right) \tag{9}$$

and this is probably good up to values of y considerably larger than those for which (6) was proved to hold. For instance, if $u_0=200$, $U=800$, $y=2{,}000$, then $-\Delta x=96$ feet, as figured from (9), is likely to be a fair correction; but if $u_0=100$ it is doubtful whether (9) would be good up to values of y as great as 2,000.

VII. A FOURTH APPROXIMATION.

One way in which to get an estimate of the true trajectory is to shut it in between two curves, one above, the other below it. Clearly $x<s<x+y$. When x is small the relative approach of s to x is close; when s is larger the relative approach to $x+y$ is fairly good. Moreover, $y<s$.

The following three differential equations are therefore suggestive as throwing light on the trajectory

$$\frac{d^2y}{dx^2}=\frac{g}{u_o^2}\, e^{2gx/U^2}, \text{ lying too high,} \tag{10}$$

$$\frac{d^2y}{dx^2}=\frac{g}{u_o^2}\, e^{2gy/U^2}, \text{ lying too high,} \tag{11}$$

$$\frac{d^2y}{dx^2}=\frac{g}{u_o^2}\, e^{2g(x+y)/U^2}, \text{ lying too low,} \tag{12}$$

This first is immediately integrable to give

$$y=\frac{U^4}{4gu_o^2}\left(e^{2gx/U^2}-1\right)-\frac{U^2x}{2u_o^2} \tag{13}$$

This curve leads to the approximation (5) as a special case. It has, however, but small relation to the general features of a long trajectory. The broad fundamental feature of a long trajectory is that the projectile drops vertically at the end, i. e., the trajectory has an asymptote. Obviously, (13) shows no asymptote; every value of x yields a finite value of y.

VIII. A FIFTH APPROXIMATION.

The second equation may be integrated by the usual device of introducing the variable $p=dy/dx$, $d^2y/dx^2=pdp/dy$.

The result is

$$\cos^{-1}e^{-gy/U^2}=\frac{gx}{u_oU} \text{ or } y=\frac{U^2}{g}\log\sec\frac{gx}{u_oU}. \tag{14}$$

This curve, which, like (13), lies above the true trajectory, is a far better approximation when y is large and the curve is nearly vertical. The curve shows an asymptote at $x=\pi u_oU/2g$. The true trajectory can not have an asymptote any farther from the origin. If $u_o=100$ and $U=800$, the total limiting forward travel of the bomb can not therefore be more than 4,000 feet. The asymptotic distance varies directly with both u_o and U. It is in the case of a light object, such as a tennis ball, that the existence of the asymptote is most easily observed.

The correction for forward carry from the parabola to (14) is an underestimate because the true trajectory lies below (14) and consequently nearer the y axis. This correction is

$$-\Delta x=u_o\sqrt{\frac{2y}{g}}-\frac{u_oU}{g}\cos^{-1}e^{-gy/U^2} \tag{15}$$

If $u_o=200$, $U=800$, $y=2{,}000$, as in the previous illustrative case, the correction (an underestimate) is 38 feet as compared with 96, the accuracy of which is unknown. The very different forms of (9) and (15) as functions of u_o are noteworthy.

If $y=2{,}000$, the value of x from (14) is 2,199. For this value of x the value of y in (13), which also undercorrects the parabola, is easiest found from the series

$$y=\frac{U^4}{8gu_o^2}\left[\left(\frac{2gx}{U^2}\right)^2+1/3\left(\frac{2gx}{U^2}\right)^3+1/12\left(\frac{2gx}{U^2}\right)^4+\text{----}\right],$$

which avoids the subtraction of nearly equal large numbers. The value of y is 2,082; hence (13), which lies too high, lies lower than (14) by 82 feet in this case and gives a better trajectory. This might have been expected from the fact that when y is not too large relative to x, s is much nearer to x than to y and (10) must lie nearer the true trajectory than (11).

Now the correction for carry may be found for (13) in this manner. This curve lies 82 feet lower than (14) when $x=2{,}199$ and its slope is nearly 2. The correction from (14) to (13) is therefore 41 feet, which, added to 38, gives 79 as the correction from the parabola as compared with 96 from (9).

IX. A SIXTH APPROXIMATION.

Finally the equation of the curve (12), which is an overestimation, may be integrated. Let

$$z = x + y \frac{dz}{dx} = \frac{dy}{dx} \pm 1, \quad \frac{d^2z}{dx^2} = \frac{d^2y}{dx^2},$$

$$\frac{d^2z}{dx^2} = \frac{g}{u_o^2} e^{2gz/U^2}, \quad \left(\frac{dz}{dx}\right)^2 = \frac{U^2}{u_o^2} e^{2gz/U^2} + C.$$

When $x=0$, $y=0$, $z=0$, $dy/dx=0$, then $dz/dx=1$ and $C=1-U^2/u_o^2$.

$$\frac{dz}{x} = \sqrt{\frac{U^2}{u_o^2} e^{2gz/U^2} - \frac{U^2}{u_o^2} + 1},$$

$$\cos^{-1}\left(e^{-gz/U^2}\sqrt{1-\frac{u_o^2}{U^2}}\right) - \cos^{-1}\sqrt{1-\frac{u_o^2}{U^2}} = \frac{gx}{u_o U}\sqrt{1-\frac{u_o^2}{U^2}}, \tag{15}$$

$$y + x = \frac{U^2}{g}\left[\log \sec\left(\frac{gx}{u_o U}\sqrt{1-\frac{u_o^2}{U^2}} + \sin^{-1}\frac{u_o}{U}\right) + \log \sqrt{1-\frac{u_o^2}{U^2}}\right]. \tag{16}$$

This curve, too, shows an asymptote which, when u_o/U is small enough so that its square may be neglected, is in the position

$$x = \frac{u_o U}{g}\left[\frac{\pi}{2} - \frac{u_o}{U}\right]$$

and thus lies nearer the y axis by the absolute amount u_o^2/g. If $u_o=100$ and $U=800$, as before, the new position of the asymptote falls 310 feet short of the old position. Hence a bomb of terminal velocity 800 ft./sec. launched horizontally from an indefinite height (in an atmosphere of constant density) would have a forward travel approaching some limit between 3,700 and 4,000 feet, approximately.

To return to the case of $u_o=200$, $U=800$, $y=2,000$, the best value thus far obtained for x is 2,158 on (13)—an overestimate of x as compared with the true trajectory. Substitute this value in (16). Then $y=2,058$ and exceeds 2,000 by 58 feet. As the slope is about 2, this means an additional correction of about 30 feet to $x=2,128$. The result is that x lies between 2,128 and 2,158 feet under these conditions. The approximation (9) gave the value $x=2,141$—almost the mean, but this may be accidental.

X. SUMMARY.

It has been shown that when a bomb is launched from an airplane the velocity of 800 ft./sec. will not be attained before the bomb has fallen a distance practically equal to that required for attaining the same velocity in vertical fall from rest. Formulas (6) and (9) have been derived for the forward carry (or its correction related to the parabola) in case the vertical fall is only a few hundred feet, but neither formula can be expected to apply when y is larger than x. Three approximate trajectories (13), (14), (16) have been derived. The true trajectory lies below (13) and (14) and above (16). Curves (14) and (16) resemble the true trajectory in showing a vertical asymptote, but until y considerably exceeds x the non-asymptotic form (13) is a better approximation than (14). For an initial velocity as high as 200 ft./sec. and a terminal velocity as low as 800 ft./sec. the correction from the parabola is not great (about 100 ft.) in a drop of 2,000 feet, and the correction is known to within about 15 feet.

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